JOURNAL OF COMPUTATIONAL PHYSICS 36, 128–132 (1980)

Note

Spherical Bessel Functions of Large Order

This note introduces functions $b_n(x)$, related to spherical Bessel functions $j_n(x)$ and $y_n(x)$. They are scaled so that they are bounded functions of n and polynomially bounded functions of x, and therefore avoid the problems of underflow and overflow which are so common with Bessel functions. They can be generated from a stable recurrence relation for which starting values are readily computable.

The definitions of the special functions are well suited to classical analysis, but often not to computation. For example, the spherical Bessel functions $j_n(x)$ and $y_n(x)$ for sufficiently large order n and fixed argument x will, respectively, underflow and overflow the range of any computer, and this can be a serious embarrassment. When underflow occurs the value $\hat{j}_n(x)$ assigned to $j_n(x)$ will be zero, and when overflow occurs the value $\hat{y}_n(x)$ assigned to $y_n(x)$ will depend upon the machine, but generally will be meaningless. In either case,

$$\hat{j}_n(x)\,\hat{y}_n(x) \neq j_n(x)\,y_n(x) = -(1/(2n+1)\,x)[1 + O(n^{-1})].$$

To illustrate this point, consider the series

$$g(\mathbf{x}, \mathbf{x}') = ik \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\theta)\, h_n^{(1)}(kr)\, h_n^{(1)}(kr')\, j_n(ka)/h_n^{(1)}(ka),$$

where

$$r = |\mathbf{x}|, \qquad r' = |\mathbf{x}'|,$$

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{x}'}{rr'},$$

and

$$h_n^{(1)} = j_n + iy_n .$$

The function

$$\frac{\exp ik\,|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|} - g(\mathbf{x}, \mathbf{x}')$$

is a fundamental solution of Helmholtz's equation which vanishes on the surface of a sphere of radius a. When both $\mathbf{x}$ and $\mathbf{x}'$ are close to the surface of the sphere, this series converges very slowly. In fact,

$$h_n^{(1)}(kr)\, h_n^{(1)}(kr')\, j_n(ka)/h_n^{(1)}(ka) = -\frac{i}{(2n+1)\,ka}\left(\frac{a^2}{rr'}\right)^{n+1} [1 + O(n^{-1})] \tag{1}$$

0021-9991/80/070128-05$02.00/0

and

$$P_n(\cos\theta) = \frac{\cos((n+\frac{1}{2})\,\theta - \frac{1}{4}\pi)}{n^{1/2}(\frac{1}{2}\pi \sin\theta)^{1/2}}\,[1 + O(n^{-1})],$$

so the geometric decay begins slowly when (a^2/rr') is close to unity and terms with large n can be significant. Again, if

$$ka = kr = 1, \qquad kr' = 1 \cdot 1, \qquad n = 30,$$

then

$$\frac{1}{(2n+1)\,ka}\left(\frac{a^2}{rr'}\right)^{n+1} \simeq 8 \cdot 54 \times 10^{-4},$$

which is not insignificant. However,

$$j_n(ka) \simeq 5 \cdot 57 \times 10^{-43},$$

which will underflow the range of many machines, so the left-hand side of (1) cannot be computed by evaluating the Bessel functions and then forming the products and ratios indicated.

The remedy for this difficulty is obvious. We must extract from j_n a factor which decays with n, and from y_n a factor which grows with n, so that the residue in each case is comparable with unity. These decaying and growing factors can then be cancelled analytically from the formulas.

The purpose of this note is to introduce a scaled Bessel function $b_n(x)$, defined by

$$\begin{aligned} j_n(x) &= +\frac{1}{2}\pi^{1/2}\left(\frac{x}{2}\right)^{n}\frac{1}{\Gamma(n+\frac{3}{2})}\,b_{-n-1}(x), \\ y_n(x) &= -\frac{1}{2}\pi^{-1/2}\left(\frac{x}{2}\right)^{-n-1}\Gamma\left(n+\frac{1}{2}\right) b_n(x), \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots,$$

and to show that it is an easily computable, polynomially bounded function of n and x. In particular, when x is fixed and $n \gtrsim x$, then $b_{-n-1}(x), b_{-n}(x), \ldots, b_n(x)$ can be generated from a stable recurrence relation for which starting values are readily computable. Consequently, a device such as J. C. P. Miller's method (described in [1]) for computing spherical Bessel functions is not needed for the scaled function. It would be a simple matter to modify the definitions of the other special functions similarly.

From the definition of the Bessel function and an inequality in Watson's treatise [2], we find the series

$$b_n(x) = \sum_{k=0}^{\infty} (x/2)^{2k}\,\frac{\Gamma(n+\frac{1}{2}-k)}{k!\,\Gamma(n+\frac{1}{2})},$$

which converges uniformly and absolutely for all (n, x), and the inequalities

$$\begin{aligned} |b_n(x)| &\leqslant C_n |x|^n, & n \geqslant 0, \\ &\leqslant 1, & n < 0. \end{aligned}$$

When x is fixed and $|n| \to \infty$,

$$b_n(x) = 1 + \frac{(x/2)^2}{(n - \frac{1}{2})} + O(n^{-2}),$$

so $b_n(x)$ converges to 1, from above when $n > 0$ and from below when $n \leqslant 0$.

Then function b_k satisfies the recurrence relation

$$b_{k+1}(x) = b_k(x) - a_k(x)\, b_{k-1}(x), \tag{2}$$

where

$$a_k(x) = \frac{(x/2)^2}{k^2 - \frac{1}{4}},$$

which will be shown to be stable in the direction of increasing k. The series for $b_{-n-1}(x)$ and $b_{-n}(x)$ converge very rapidly when $n \gtrsim x$, and $b_0(x)$ and $b_1(x)$ are given by elementary functions,

$$b_0(x) = \cos x, \qquad b_1(x) = \cos x + x \sin x.$$

Thus, initial data for (2) are computable so

$$b_{-n-1}(x), b_{-n}(x), \ldots, b_{-1}(x)$$

and

$$b_0(x), b_1(x), \ldots, b_n(x)$$

can be generated by recurrence.

In order to investigate the stability of the recurrence relation for b_k, set

$$\mathbf{b}_k = \begin{pmatrix} b_{k-1} \\ b_k \end{pmatrix},$$

so that

$$\mathbf{b}_{k+1} = A_k \mathbf{b}_k, \tag{3}$$

where

$$A_k = \begin{pmatrix} 0 & 1 \\ -a_k & 1 \end{pmatrix}.$$

The computed solution $\hat{\mathbf{b}}_k$ will not satisfy (3) but rather

$$\hat{\mathbf{b}}_{k+1} = A_k \hat{\mathbf{b}}_k + \mathbf{e}_k,$$

where $\mathbf{e}_k$ denotes the error introduced by rounding at the kth iteration of (3). The cumulative error,

$$\mathbf{r}_k = \hat{\mathbf{b}}_k - \mathbf{b}_k ,$$

will also satisfy

$$\mathbf{r}_{k+1} = A_k \mathbf{r}_k + \mathbf{e}_k$$

with initial data equal to the error in the computed initial data for (3). The eigenvalues of A_k are

$$\lambda_k^{\pm} = \tfrac{1}{2}(1 \pm (1 - 4a_k)^{1/2}).$$

which have the following properties:

(1) $\lambda_k^{\pm} = \frac{1}{2} \pm \mu$, $k = 0$, where $\mu = (x^2 + \frac{1}{4})^{1/2}$;

(2) $\lambda_k^{+} = \overline{\lambda_k^{-}}$, $| \lambda_k^{\pm} | = a_k^{1/2}$, $0 < | k | \leqslant \mu$;

(3) $0 < \lambda_l^{-} < \lambda_k^{-} < \frac{1}{2} < \lambda_k^{+} < \lambda_l^{+} < 1, \mu < | k | < | l |$.

Let

$$P_k = \begin{pmatrix} 1 & 1 \\ \lambda_k^{-} & \lambda_k^{+} \end{pmatrix},$$

so that

$$A_k = P_k J_k P_k^{-1},$$

where

$$J_k = \begin{pmatrix} \lambda_k^{-} & 0 \\ 0 & \lambda_k^{+} \end{pmatrix}.$$

Introduce the norms

$$\| \mathbf{v} \|_k = \| P_k^{-1} \mathbf{v} \|_\infty , \qquad \mathbf{v} \in C^2,$$

with respect to which

$$\| A_k \|_k = \rho(A_k) = \max\{| \lambda_k^{+} | , | \lambda_k^{-} |\},$$

where $\rho(A_k)$ denotes the spectral radius of A_k. Note that these definitions fail if $| k | = \mu$ and the eigenvalues of A_k coincide, but the modifications required are trivial so this case will be dismissed. The norms are compatible, since

$$\begin{aligned} \| \mathbf{v} \|_k &= \| P_k^{-1} P_{k-1} P_{k-1}^{-1} \mathbf{v} \|_\infty \\ &\leqslant \| P_k^{-1} P_{k-1} \|_\infty \| v \|_{k-1} , \end{aligned}$$

and

$$\begin{aligned} \| \mathbf{r}_{k+1} \|_k &\leqslant \| \mathbf{e}_k \|_k + \| A_k \|_k \| \mathbf{r}_k \|_k \\ &\leqslant \| \mathbf{e}_k \|_k + \| A_k \|_k \| P_k^{-1} P_{k-1} \|_\infty \| \mathbf{r}_k \|_{k-1} . \end{aligned}$$

In order that the recurrence relation should be stable, the errors must not grow geometrically, and to secure this we require that

$$p_k = \| A_k \|_k \, \| P_k^{-1} P_{k-1} \|_\infty < 1,$$

ideally for all k but in practice only whenever $|k|$ is large. Now it is easy to verify that

$$\begin{aligned} \| P_k^{-1} P_{k-1} \|_\infty &= 1, & k &> 1 + \mu, \\ &= (\lambda_{k-1}^+ - \lambda_{k-1}^-)/(\lambda_k^+ - \lambda_k^-) > 1, & k &< -\mu, \end{aligned}$$

so

$$\begin{aligned} p_k &= \lambda_k^+ < 1, & k &> 1 + \mu, \\ &= \lambda_k^+(\lambda_{k-1}^+ - \lambda_{k-1}^-)/(\lambda_k^+ - \lambda_k^-), & k &< -\mu. \end{aligned}$$

A somewhat longer calculation shows that

$$\lambda_k^+(\lambda_{k-1}^+ - \lambda_{k-1}^-)/(\lambda_k^+ - \lambda_k^-) < 1$$

if $k < -\mu$ and k also satisfies the inequality

$$\begin{aligned} 8k^6 &- (15x^2 + 38)\, k^4 + (12x^2 + 32)\, k^3 + (6x^4 + \tfrac{75}{2}x^2 + \tfrac{3}{2})\, k^2 \\ &- (12x^4 + 39x^2 + 8)\, k + (x^6 + \tfrac{9}{2}x^4 + 9\tfrac{9}{16}x^2 + \tfrac{15}{8}) > 0. \end{aligned} \tag{4}$$

Note that (4) certainly will hold if

$$k < -(|x| + 4).$$

Thus, the stability of the recurrence scheme is proven.

REFERENCES

1. M. ABRAMOWITZ AND I. A. STEGUN, "Handbook of Mathematical Functions," p. 452, Dover, New York, 1965.
2. G. N. WATSON, "A Treatise on the Bessel Functions," p. 49, Cambridge Univ. Press, Cambridge, England, paperback edition, 1966.

RECEIVED: April 17, 1979

D. M. O'BRIEN
Department of Mathematical Physics
University of Adelaide
G.P.O. Box 498
Adelaide, South Australia 5001